Remark. In each of the following examples, we need to find $\delta > 0$ (when ε is given) such that certain estimates

$$\frac{|f_1(x)|}{|f_2(x)|}|x - x_0| < \varepsilon \quad \forall\, x \in V_\delta(x_0) \setminus \{x_0\}.$$

To do this, one first try to select suitable $\delta_1 > 0$, $\delta_2 > 0$ and $M_1, M_2 > 0$ such that

$$|f_1(x)| \le M_1 \quad \text{whenever } 0 < |x - x_0| < \delta_1$$

and

$$M_2 \le |f_2(x)| \quad \text{- - - - - - - - -} \quad \delta_2.$$

In this case, one has

$$\frac{|f_1(x)|}{|f_2(x)|}|x - x_0| \le \frac{M_1}{M_2}|x - x_0| \quad \text{whenever } 0 < |x - x_0| < \delta_1 \wedge$$

Thus, we are led to take $\delta > 0$ s.t. $\delta \le \delta_1 \wedge \delta_2$

and $\delta \le \varepsilon \cdot \dfrac{M_2}{M_1}$ because then, whenever $0 < |x - x_0| < \delta$, one has

$$\frac{|f_1(x)|}{|f_2(x)|}|x - x_0| \le \frac{M_1}{M_2}|x - x_0| < \frac{M_1}{M_2}\delta \le \frac{M_1}{M_2}\left(\frac{M_2}{M_1}\varepsilon\right) = \varepsilon$$

Examples (for limits of functions), done in ε-δ terminology.

1. $\lim_{x \to x_0} x^2 = x_0^2$. Let $\varepsilon > 0$. Take δ s.t. $\delta := \min\left\{1, \frac{\varepsilon}{2|x_0|+1}\right\}$ (>

Then, whenever $0 < |x - x_0| < \delta$, one has

$$|x| \leq |x - x_0| + |x_0| < \delta + |x_0| \leq 1 + |x_0|$$

and so $|x^2 - x_0| < \varepsilon$ because

$$|x^2 - x_0^2| = |x - x_0|\,|x + x_0| \leq |x - x_0| \left(|x| + |x_0|\right) \leq (2|x_0| + 1)\,|x - x_0$$

$$< (2|x_0|+1)\,\delta \leq (2|x_0|+1) \cdot \frac{\varepsilon}{2|x_0|+1} = \varepsilon$$

2. $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$ $\forall x_0 \neq 0$. Let $\varepsilon > 0$. Take $\delta > 0$ s.t.

$$\delta := \min\left\{\frac{|x_0|}{2}, \frac{|x_0|^2}{2}\varepsilon\right\}$$

Then, whenever $0 < |x - x_0| < \delta$, one has

$|x_0| - |x| \leq |x - x_0| < \delta \leq \frac{|x_0|}{2}$ (so $\frac{|x_0|}{2} < |x|$), and

so $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$ because

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{|x| \cdot |x_0|} \leq \frac{|x_0 - x|}{\frac{|x_0|}{2} \cdot |x_0|} < \frac{2}{|x_0|^2} \cdot \delta \leq \frac{2}{|x_0|^2}\left(\frac{|x_0|^2}{2}\varepsilon\right) \leq$$

3. $\lim_{x \to 2} \frac{x^3+1}{x^2-1} = 3$. Let $\varepsilon > 0$. Take $\delta > 0$ s.t.

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{14}\right\}.$$

Then, whenever $0 < |x - 2| < \delta$, one has $0 < |x - 2| < \frac{1}{2}$ so

$$\frac{3}{2} < 2 - \frac{1}{2} < x < 2 + \frac{1}{2} < 3$$

hence $1 < \left(\frac{3}{2}\right)^2 - 1 < x^2 - 1$ and $|x| < 3$, and consequent

$$\left|\frac{x^3+1}{x^2-1} - 3\right| = \frac{|(x^3+1) - 3(x^2-1)|}{|x^2-1|} \leq |(x^3+1) - 3(x^2-1)| = |x-2|\,|x^2 - x + 2$$

$$\leq |x-2| \cdot (|x|^2 + |x| + 2) \leq |x-2|\,(3^2 + 3 + 2) < 14\delta \leq \varepsilon$$